

In each case the time of charging was two minutes.

The results are given in terms of the total charge in electromagnetic units contained in the specimen when the original difference of potential between the surfaces was unity. In the first column times measured from the instant at which both surfaces are connected to the earth are given.

These results are plotted as figs. 20-22. For these specimens the values of α are (after two minutes charging):—

Glass (test-tube).	Paraffin.	Ebonite.
0·0115	0·0048	0·0055

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On Deep-Sea Water Waves caused by a Local Disturbance on or beneath the Surface.

By K. TERAZAWA, Ri-Gakushi in the Imperial University of Tokyo.

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The classical problem of waves produced in deep-sea water by a local disturbance of the free surface has been investigated by Prof. H. Lamb* in a very able manner. He completed the theory of wave propagation in one dimension and in two horizontal dimensions when the initial disturbance is concentrated in the immediate neighbourhood of a line or a point, assuming that Fourier's double integral theorem can be applied in such a case. In this paper I venture to discuss the same problem, especially the oscillations at the centre of the disturbance, following Prof. Lamb's method, in cases where the initial disturbance is spread over a certain extent of the free surface; and, as an application of the general solution, I propose to treat the case in which the initial disturbing source is situated at a finite depth from the free surface, *i.e.* where the surface wave is produced by an explosion like that of a mine under water.

§ 1. Supposing the water to extend to infinity, horizontally and downwards, and taking the axes of x and y on the undisturbed free surface and that of z

* 'Lond. Math. Soc. Proc.,' vol. 2 (2), p. 371 (1904), or 'Hydrodynamics,' third edition, §§ 236-239, 251-252.

vertically downwards, we have, on the assumption that the motion is infinitely small and irrotational,

$$p/\rho = \partial\phi/\partial t + gz, \quad (1)$$

where p denotes the pressure, ρ the density of water, g the constant of gravity and ϕ the velocity-potential which satisfies the equation

$$\nabla^2\phi = 0. \quad (2)$$

If ζ denotes the depression of water-level at the point $(x, y, 0)$ and at time t below the undisturbed surface, then the pressure condition to be satisfied at the free surface, supposing ζ to be small, is

$$\zeta = -\frac{1}{g} \left[\frac{\partial\phi}{\partial t} \right]_{z=0}, \quad (3)$$

and the kinematical condition is

$$\frac{\partial\zeta}{\partial t} = - \left[\frac{\partial\phi}{\partial z} \right]_{z=0}. \quad (4)$$

Hence, for $z = 0$, we must have

$$\frac{\partial^2\phi}{\partial t^2} = g \frac{\partial\phi}{\partial z},$$

or, if the time factor be $e^{i(\sigma t + \epsilon)}$,

$$\sigma^2\phi = -g \frac{\partial\phi}{\partial z}. \quad (5)$$

The solution of equation (2), which makes $\phi = 0$ when $z = \infty$, is, in cylindrical co-ordinates (ϖ, θ, z) ,

$$\phi = e^{-kz} J_m(k\varpi) \left. \begin{matrix} \cos \\ \sin \end{matrix} \right\} m\theta, \quad (6)$$

where J_m is Bessel's function of m th order and k is any positive number.

From equations (5) and (6) we get

$$\sigma^2 = gk. \quad (7)$$

The modes of irrotational motion of water by surface disturbance present themselves, as Prof. Lamb points out, in two forms: (1) by an initial displacement of the surface, without initial velocity; (2) by an initial impulse applied on the surface, without initial surface displacement. The most general case is a combination of these two.

I.

§ 2. As the typical solution for the case of initial rest we take

$$\phi = g \frac{\sin\sigma t}{\sigma} e^{-kz} J_m(k\varpi) \cos m\theta, \quad (8)$$

$$\zeta = -\cos\sigma t J_m(k\varpi) \cos m\theta. \quad (9)$$

The water is supposed to be unlimited in extent, and hence there is no limitation to the length of waves, but the motion will be the result of the superposition of waves of infinite variety of length. Therefore to obtain the general expression which embraces the superposition of all such solutions we must make use of the double integral theorem of Fourier's type

$$f(\varpi) = \int_0^\infty J_m(k\varpi) k dk \int_0^\infty f(\alpha) J_m(k\alpha) \alpha d\alpha, \quad (10)$$

with some conditions concerning the function $f(\varpi)$.

Thus, corresponding to the initial conditions,

$$\zeta = -f(\varpi) \cos m\theta, \quad \phi = 0, \quad (11)$$

we have

$$\phi = g \cos m\theta \int_0^\infty \Phi(k) \frac{\sin \sigma t}{\sigma} e^{-kz} J_m(k\varpi) k dk, \quad (12)$$

$$\zeta = -\cos m\theta \int_0^\infty \Phi(k) \cos \sigma t J_m(k\varpi) k dk, \quad (13)$$

where

$$\Phi(k) = \int_0^\infty f(\alpha) J_m(k\alpha) \alpha d\alpha; \quad (14)$$

for these expressions clearly satisfy the conditions specified above. As to the disturbing function $f(\varpi)$, we can assume its form in several ways, under the condition that the integral theorem (10) must remain valid.

Now we consider the case where the initial elevation of the surface is spread over the whole surface according to the law

$$f(\varpi) = \frac{A}{\varpi}. \quad (15)$$

This is the simplest form which satisfies the double integral theorem. By taking the value of A sufficiently small, we may consider that it approaches, when $m = 0$, to the case where the initial displacement of great amount is confined to the neighbourhood of the origin.*

[* It has been kindly pointed out to me by a referee that this statement requires correction and precision. However small A may be taken, the initial elevation is spread uniformly along the radius, but is not concentrated in the sense of Cauchy and Poisson, though it is to be expected that only the central part when the height is considerable can contribute sensibly to the amplitudes of the waves. It might appear that the total initial elevated volume, which is expressed by the integral $2\pi \int_0^\infty f(\varpi) \varpi d\varpi$ when $m = 0$, should be finite. But it seems to me that this condition is not a necessary one for the validity of the integral theorem (10). This remark applies to both the initial distribution A/ϖ and $A/(b^2 + \varpi^2)^{\frac{1}{2}}$ that is treated in § 4. But for the initial distribution $A b/(b^2 + \varpi^2)^{\frac{3}{2}}$ that is treated on p. 69 there can be no doubt, for it is of finite total amount ;

Put the expression (15) into (14), then it appears, by a well known formula, that

$$\Phi(k) = A \int_0^\infty J_m(k\alpha) d\alpha = A/k. * \quad (16)$$

Whence the generalised expressions for ϕ and ζ will become

$$\phi = Ag \cos m\theta \int_0^\infty \frac{\sin \sigma t}{\sigma} e^{-kz} J_m(k\varpi) dk, \quad (17)$$

$$\zeta = -A \cos m\theta \left[\int_0^\infty \cos \sigma t e^{-kz} J_m(k\varpi) dk \right]_{z=0}. \quad (18)$$

By expanding $\sin \sigma t$ and $\cos \sigma t$ into series, we have

$$\phi = Agt \cos m\theta \sum_{n=0}^\infty (-1)^n \frac{(gt^2)^n}{(2n+1)!} \int_0^\infty k^n e^{-kz} J_m(k\varpi) dk, \quad (19)$$

$$\zeta = -A \cos m\theta \sum_{n=0}^\infty (-1)^n \frac{(gt^2)^n}{2n!} \left[\int_0^\infty k^n e^{-kz} J_m(k\varpi) dk \right]_{z=0}; \quad (20)$$

here σ is eliminated by the relation $\sigma^2 = gk$.

The definite integral contained in (19) and (20) can be evaluated as follows:—

By using Hansen's form of Bessel's function

$$J_m(k\varpi) = \frac{i^m}{\pi} \int_0^\pi e^{-ik\varpi \cos \phi} \cos m\phi d\phi,$$

and the integral formula

$$\int_0^\infty k^n e^{-k(z+i\varpi \cos \phi)} dk = \frac{n!}{(z+i\varpi \cos \phi)^{n+1}}, \quad (z > 0),$$

we find that

$$\int_0^\infty k^n e^{-kz} J_m(k\varpi) dk = \frac{i^m n!}{\pi} \int_0^\pi \frac{\cos m\phi d\phi}{(z+i\varpi \cos \phi)^{n+1}}.$$

If we put

$$r = \sqrt{(z^2 + \varpi^2)}, \quad \mu = z/r, \quad (21)$$

then we have

$$\int_0^\infty k^n e^{-kz} J_m(k\varpi) dk = \frac{i^m n!}{\pi r^{n+1}} \int_0^\pi \frac{\cos m\phi d\phi}{[\mu + (\mu^2 - 1)^{\frac{1}{2}} \cos \phi]^{n+1}}.$$

and when b vanishes it is concentrated at a point, the result being then the well known form of Cauchy and Poisson.

A main object of this paper is, as stated, to trace the effect on the waves propagated to a distance, and especially on the course of the disturbance at the centre, of a broadening of the initial disturbance from a mere point to a wider region. This has also the advantage that we do not have to proceed to a limit.]

* Gray and Mathews, 'Treatise on Bessel Functions,' p. 79.

Now, define the function $P_n^{-m}(\mu)$ by

$$P_n^{-m}(\mu) = (1-\mu^2)^{-m/2} \int_{\mu}^1 \int_{\mu}^1 \dots P_n(\mu) d\mu^m,$$

$P_n(\mu)$ being the zonal harmonic of n th order, then the integral expression of $P_n^{-m}(\mu)$, which corresponds to Laplace's integral for $P_n(\mu)$, will be

$$P_n^{-m}(\mu) = \frac{i^m n!}{\pi(n+m)!} \int_0^\pi \frac{\cos m\phi d\phi}{[\mu + (\mu^2 - 1)^{\frac{1}{2}} \cos \phi]^{n+1}}.$$

Therefore the required integral becomes

$$\int_0^\infty k^n e^{-kz} J_m(k\varpi) dk = \frac{(n+m)!}{r^{n+1}} P_n^{-m}(\mu). \quad (22)$$

The result of this integral, in the case of $n \geq m$, was found by Prof. E. W. Hobson.* In this case, if we remember that

$$P_n^{-m}(\mu) = \frac{(n-m)!}{(n+m)!} P_n^m(\mu),$$

where $P_n^m(\mu)$ is the associated function defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m},$$

we have

$$\int_0^\infty k^n e^{-kz} J_m(k\varpi) dk = \frac{(n-m)!}{r^{n+1}} P_n^m(\mu), \quad (n \geq m) \quad (23)$$

in agreement with his result. Hereafter we will use (22) for $n < m$ and (23) for $n \geq m$.

Hence, putting these values (22) and (23) into equations (19) and (20), we get

$$\begin{aligned} \phi = \frac{Agt \cos m\theta}{r} \left\{ \sum_{n=0}^{m-1} (-1)^n \frac{(n+m)!}{(2n+1)!} \left(\frac{gt^2}{r} \right)^n P_n^{-m}(\mu) \right. \\ \left. + \sum_{n=m}^{\infty} (-1)^n \frac{(n-m)!}{(2n+1)!} \left(\frac{gt^2}{r} \right)^n P_n^m(\mu) \right\}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \zeta = - \frac{A \cos m\theta}{\varpi} \left\{ \sum_{n=0}^{m-1} (-1)^n \frac{(n+m)!}{2n!} \left(\frac{gt^2}{\varpi} \right)^n P_n^{-m}(0) \right. \\ \left. + \sum_{n=m}^{\infty} (-1)^n \frac{(n-m)!}{2n!} \left(\frac{gt^2}{\varpi} \right)^n P_n^m(0) \right\}. \end{aligned} \quad (25)$$

* 'Lond. Math. Soc. Proc.', vol. 25, p. 73.

The values of $P_n^{-m}(0)$ and $P_n^m(0)$ are as follows:—

$$\left. \begin{aligned} P_n^{-m}(0) &= \frac{(m+n-1)(m+n-3)\dots(m-n+1)}{(m+n)!}, & (n > m) \\ &= \frac{1}{m!}, & (n=0) \\ P_n^m(0) &= (-1)^{(n-m)/2} \frac{1 \cdot 3 \dots (n+m-1)}{2 \cdot 4 \dots (n-m)}, & (n-m \text{ even}) \\ &= 0, & (n-m \text{ odd}) \\ &= 1 \cdot 3 \dots (2n-1), & (n=m) \end{aligned} \right\} \quad (26)$$

For the case of symmetry about the origin, *i.e.* $m=0$, the first summations in (24) and (25) disappear, and therefore we have simply

$$\phi_{m=0} = \frac{Agt}{r} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n+1)!} \left(\frac{gt^2}{r}\right)^n P_n(\mu), \quad (27)$$

$$\zeta_{m=0} = -\frac{A}{\varpi} \left\{ 1 - \frac{1^2}{4!} \left(\frac{gt^2}{\varpi}\right)^2 + \frac{1^2 \cdot 3^2}{8!} \left(\frac{gt^2}{\varpi}\right)^4 - \dots \right\}, \quad (28)$$

similar expressions to those which were found by Prof. Lamb for the case of complete concentration of the source.

The next simplest case is that of $m=1$, though it may not be very important in practice. In this case we obtain

$$\phi_{m=1} = \frac{Agt \cos \theta}{r} \left\{ P_0^{-1}(\mu) + \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{(2n+1)!} \left(\frac{gt^2}{r}\right)^n P_n^1(\mu) \right\}, \quad (29)$$

$$\zeta_{m=1} = -\frac{A \cos \theta}{\varpi} \left\{ 1 - \frac{1}{2!} \left(\frac{gt^2}{\varpi}\right) + \frac{1^2 \cdot 3}{6!} \left(\frac{gt^2}{\varpi}\right)^3 - \frac{1^2 \cdot 3^2 \cdot 5}{10!} \left(\frac{gt^2}{\varpi}\right)^5 + \dots \right\}, \quad (30)$$

These solutions do not give any information as to what takes place at the immediate neighbourhood of the origin. Not only the initial data but the displacement at any time will be infinite at the origin; such a point is excluded in our fundamental assumption. The appropriate initial data and the solutions to illustrate this point will be given presently (§ 4).

§ 3. The series (25) is not convenient when we deal with the case in which gt^2/ϖ has large values, since it converges rather slowly in such a case. The suitable expressions in this case, at least when $m=0$ and $m=1$, can be obtained by a similar method to that employed by Prof. Lamb for the case of complete concentration of the source.

If we put

$$T_m(z) = \int_0^{\infty} \cos \sigma t e^{-kz} J_m(k\varpi) dk, \quad (31)$$

then returning to (18) we have

$$\zeta = -A \cos m\theta \cdot T_m(0). \quad (32)$$

By using the integral expression for J_m

$$J_m(k\varpi) = \frac{1}{\pi} \int_0^\pi \cos(m\phi - k\varpi \sin \phi) d\phi,$$

and putting

$$C(z) = \int_0^\infty e^{-kz} \cos \sigma t \cos(k\varpi \sin \phi) dk,$$

$$S(z) = \int_0^\infty e^{-kz} \cos \sigma t \sin(k\varpi \sin \phi) dk,$$

with

$$\sigma^2 = gk;$$

we get

$$T_m(z) = \frac{1}{\pi} \int_0^\pi C(z) \cos m\phi d\phi + \frac{1}{\pi} \int_0^\pi S(z) \sin m\phi d\phi. \quad (33)$$

Now expanding $\cos \sigma t$ into a series and using the formulæ

$$\left. \begin{aligned} \int_0^\infty k^n e^{-kz} \cos xk dk &= \frac{n!}{p^{n+1}} \cos(n+1)\psi \\ \int_0^\infty k^n e^{-kz} \sin xk dk &= \frac{n!}{p^{n+1}} \sin(n+1)\psi \end{aligned} \right\} \quad (34)$$

where

$$p = \sqrt{(z^2 + x^2)}, \quad \psi = \tan^{-1}(x/z)$$

and

$$x = \varpi \sin \phi;$$

we find

$$\begin{aligned} C(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{n!}{2n!p} \left(\frac{gt^2}{p} \right)^n \cos(n+1)\psi, \\ S(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{n!}{2n!p} \left(\frac{gt^2}{p} \right)^n \sin(n+1)\psi. \end{aligned}$$

$z = 0$ corresponds to $p = \varpi \sin \phi$, $\psi = \frac{1}{2}\pi$. Hence

$$\left. \begin{aligned} C(0) &= \frac{1}{\varpi \sin \phi} \left\{ \omega - \frac{\omega^3}{1.3.5} + \frac{\omega^5}{1.3.5.7.9} - \dots \right\} \\ S(0) &= \frac{1}{\varpi \sin \phi} \left\{ 1 - \frac{\omega^2}{1.3} + \frac{\omega^4}{1.3.5.7} - \dots \right\} \end{aligned} \right\} \quad (35)$$

where $\omega = gt^2/2\varpi \sin \phi$.

The functions $C(0)$ and $S(0)$ are closely connected with Fresnel's integral as is well known. If we put

$$\left. \begin{aligned} C(0) &= \frac{1}{\varpi \sin \phi} M(\omega) \\ S(0) &= \frac{1}{\varpi \sin \phi} \{1 - N(\omega)\} \end{aligned} \right\} \quad (36)$$

then we shall have*

$$\left. \begin{aligned} M(\omega) &= \frac{\sqrt{\omega}}{2} \left\{ \cos \frac{\omega}{2} \int_0^{\omega} \cos \frac{u}{2} \frac{du}{\sqrt{u}} + \sin \frac{\omega}{2} \int_0^{\omega} \sin \frac{u}{2} \frac{du}{\sqrt{u}} \right\} \\ N(\omega) &= \frac{\sqrt{\omega}}{2} \left\{ \sin \frac{\omega}{2} \int_0^{\omega} \cos \frac{u}{2} \frac{du}{\sqrt{u}} - \cos \frac{\omega}{2} \int_0^{\omega} \sin \frac{u}{2} \frac{du}{\sqrt{u}} \right\} \end{aligned} \right\} \quad (37)$$

If $gt^2/2\pi$ is a large quantity, then ω is also large, and the functions M and N can be expressed as semi-convergent series of the form†

$$\left. \begin{aligned} M(\omega) &= \frac{\sqrt{(\pi\omega)}}{2} \left\{ \sin \frac{\omega}{2} + \cos \frac{\omega}{2} \right\} - \left\{ \frac{1}{\omega} - \frac{1 \cdot 3 \cdot 5}{\omega^3} + \dots \right\} \\ N(\omega) &= \frac{\sqrt{(\pi\omega)}}{2} \left\{ \sin \frac{\omega}{2} - \cos \frac{\omega}{2} \right\} + \left\{ 1 - \frac{1 \cdot 3}{\omega^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{\omega^4} - \dots \right\} \end{aligned} \right\} \quad (38)$$

If we neglect ω^{-1} and higher powers of it in these series, we have, from (36), approximately

$$\begin{aligned} C(0) &= \frac{\sqrt{(\pi\omega)}}{2\pi \sin \phi} \left\{ \cos \frac{\omega}{2} + \sin \frac{\omega}{2} \right\}, \\ S(0) &= \frac{\sqrt{(\pi\omega)}}{2\pi \sin \phi} \left\{ \cos \frac{\omega}{2} - \sin \frac{\omega}{2} \right\}. \end{aligned}$$

Putting $\alpha = \frac{gt^2}{4\pi}$, or $\frac{\omega}{2} = \frac{\alpha}{\sin \phi}$, (39)

and

$$\left. \begin{aligned} U_m &= \int_0^{\pi} \left\{ \cos \left(\frac{\alpha}{\sin \phi} \right) + \sin \left(\frac{\alpha}{\sin \phi} \right) \right\} \frac{\cos m\phi}{(\sin \phi)^{3/2}} d\phi \\ V_m &= \int_0^{\pi} \left\{ \cos \left(\frac{\alpha}{\sin \phi} \right) - \sin \left(\frac{\alpha}{\sin \phi} \right) \right\} \frac{\sin m\phi}{(\sin \phi)^{3/2}} d\phi \end{aligned} \right\}, \quad (40)$$

$T_m(0)$ can be written in the form

$$T_m(0) = \frac{1}{\pi} \left(\frac{\alpha}{2\pi} \right)^{\frac{1}{2}} \{U_m + V_m\}. \quad (41)$$

From (40) it appears that $U_m = 0$ when m is an odd number, and $V_m = 0$ when m is an even number.

If we find the values of U_m and V_m in proper form, we can obtain the expression for ζ by (41) and (32).

The special case $m = 0$ was discussed by Prof. Lamb and it was found approximately that

$$T_0(0) = \frac{\sqrt{2}}{\pi} \cos \left(\frac{gt^2}{4\pi} \right), \quad (42)$$

* Lamb, 'Hydrodynamics,' p. 366.

† Lord Rayleigh, 'Scientific Papers,' III, p. 129.

and therefore we get

$$\zeta_{m=0} = -\frac{\sqrt{2A}}{\varpi} \cos\left(\frac{gt^2}{4\varpi}\right). \quad (43)$$

For the case $m = 1$, $U_1 = 0$,

$$V_1 = \int_0^\pi \left\{ \cos\left(\frac{\alpha}{\sin \phi}\right) - \sin\left(\frac{\alpha}{\sin \phi}\right) \right\} \frac{d\phi}{\sqrt{\sin \phi}},$$

$$\text{and} \quad T_1(0) = \frac{1}{\varpi} \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} V_1. \quad (44)$$

The value of the integral contained in the expression of V_1 cannot be found very easily. Since we see that

$$\int_0^\pi \sin\left(\frac{\alpha}{\sin \phi}\right) \frac{d\phi}{\sqrt{\sin \phi}} = 2 \int_0^{\pi/2} \sin\left(\frac{\alpha}{\cos \phi}\right) \frac{d\phi}{\sqrt{\cos \phi}} = 2 \int_1^\infty \frac{\sin \alpha u}{\sqrt{u}} \frac{du}{\sqrt{(u^2-1)}},$$

and similarly

$$\int_0^\pi \cos\left(\frac{\alpha}{\sin \phi}\right) \frac{d\phi}{\sqrt{\sin \phi}} = 2 \int_1^\infty \frac{\cos \alpha u}{\sqrt{u}} \frac{du}{\sqrt{(u^2-1)}},$$

and if we remember that

$$\frac{\sin x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}} \cdot J_{\frac{1}{2}}(x), \quad \frac{\cos x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}} \cdot J_{-\frac{1}{2}}(x),$$

then we can evaluate these integrals, following the method which is fully explained by N. Nielsen in his book on Cylinder-Functions.* I calculated these integrals after this method, and found that

$$\left. \begin{aligned} \int_1^\infty \frac{\sin \alpha u}{\sqrt{u}} \frac{du}{\sqrt{(u^2-1)}} &= -\alpha^{\frac{1}{2}}(\pi/2)^{3/2} \left\{ J_{\frac{1}{4}}^2(\alpha/2) - \sqrt{2} J_{\frac{1}{4}}(\alpha/2) J_{-\frac{1}{4}}(\alpha/2) \right\} \\ \int_1^\infty \frac{\cos \alpha u}{\sqrt{u}} \frac{du}{\sqrt{(u^2-1)}} &= \alpha^{\frac{1}{2}}(\pi/2)^{3/2} \left\{ J_{-\frac{1}{4}}^2(\alpha/2) - \sqrt{2} J_{\frac{1}{4}}(\alpha/2) J_{-\frac{1}{4}}(\alpha/2) \right\} \end{aligned} \right\} \quad (45)$$

For large values of α we can use the asymptotic expansions of Bessel's functions, and, to the degree of approximation above mentioned, it is sufficient to retain only the first term of them. Thus we may take

$$J_{\frac{1}{4}}(\alpha/2) = \frac{2}{\sqrt{(\pi\alpha)}} \cos(\alpha/2 - 3\pi/8), \quad J_{-\frac{1}{4}}(\alpha/2) = \frac{2}{\sqrt{(\pi\alpha)}} \cos(\alpha/2 - \pi/8).$$

Putting these values in the above formulæ (45), then by (44), after simplifying, we have

$$T_1(0) = -\sqrt{2/\varpi} \cdot \sin(gt^2/4\varpi). \quad (46)$$

Substituting this for $T_1(0)$ in (32), we obtain

$$\zeta_{m=1} = \sqrt{2A/\varpi} \cdot \sin(gt^2/4\varpi) \cos \theta. \quad (47)$$

* 'Handbuch d. Cylinderfunktionen,' §§ 77, 81, 82.

Though we have been considering in the above only the terms involving $\cos m\theta$, it must be understood that the same analysis could be applied to the terms containing $\sin m\theta$, if the initial data be so given.

§ 4. As a proper assumption to illustrate what occurs at the origin, take the disturbing function of such a form as

$$f(\varpi) = A/\sqrt{(b^2 + \varpi^2)} \quad (48)$$

instead of (15). This function does not violate the double integral theorem (10). Supposing the value of b to be sufficiently small, this approaches, at a distance from the origin, to the function assigned in § 2, and yet it is finite at the origin. Therefore we might take, even in this modified assumption, the solutions in § 2 as those for the point at which ϖ has a moderately large value.

For the case of symmetry ($m = 0$),* the function $\Phi(k)$ defined by (14) becomes

$$\Phi(k) = A \int_0^\infty \frac{J_0(k\alpha) \alpha d\alpha}{\sqrt{(b^2 + \alpha^2)}}.$$

This integral can be effected by using the double integral theorem (10) in a special case $m = 0$, and is found easily to be

$$\Phi(k) = Ae^{-bk}/k, \dagger \quad (49)$$

supposing b to be positive.

Hence we have the expressions for ϕ and ζ from (17), (18)

$$\phi = Ag \int_0^\infty \frac{\sin \sigma t}{\sigma} e^{-(z+b)k} J_0(k\varpi) dk, \quad (50)$$

$$\zeta = -A \int_0^\infty \cos \sigma t e^{-bk} J_0(k\varpi) dk. \quad (51)$$

Making use of the integral formula (23), we get the following solutions, in place of (27) and (28)

$$\phi = \frac{Ag t}{s} \sum_{n=0}^\infty (-1)^n \frac{n!}{(2n+1)!} \left(\frac{gt^2}{s}\right)^n P_n(\nu), \quad (52)$$

with $s = \sqrt{[(z+b)^2 + \varpi^2]}, \quad \nu = (z+b)/s,$

and $\zeta = -\frac{A}{s_0} \sum_{n=0}^\infty (-1)^n \frac{n!}{2n!} \left(\frac{gt^2}{s_0}\right)^n P_n(\nu_0), \quad (53)$

with $s_0 = \sqrt{(b^2 + \varpi^2)}, \quad \nu_0 = b/s_0.$

At the origin ($\varpi = 0$), we have

$$s_0 = b, \quad \nu_0 = 1,$$

* The integral theorem (10) is valid for $\alpha = 0$ only when $m = 0$.

† Or, as a special case of the integral of N. Sonine ('Math. Ann.,' vol. 16, p. 50), we get the above result, putting $m = 0$, $n = -\frac{1}{2}$ in his formula ω_{14} .

and since $P_n(1) = 1$, the equation (53) gives

$$\begin{aligned}\zeta_0 &= -\frac{A}{b} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{2n!} \left(\frac{gt^2}{b}\right)^n \\ &= -\frac{A}{b} \left\{ 1 - \left(\frac{gt^2}{2b}\right) + \frac{1}{1.3} \left(\frac{gt^2}{2b}\right)^2 - \frac{1}{1.3.5} \left(\frac{gt^2}{2b}\right)^3 + \dots \right\}\end{aligned}\quad (54)$$

If we put, for the present,

$$gt^2/2b = x,$$

then the above expression can, referring to (35), (36), be put in the form

$$\zeta_0 = A/b \cdot \{-1 - iM(ix) + N(ix)\},$$

and, therefore, using the integral expressions (37) for the functions M and N, we have

$$\zeta_0 = A/b \cdot \left\{ -1 + \frac{1}{2} \sqrt{x} e^{-x/2} \int_0^x e^{x/2} dx / \sqrt{x} \right\}.$$

Or, putting again, for the sake of simplicity,

$$\frac{1}{2}x = \tau^2 = gt^2/4b, \quad \zeta_0 = A\eta/b, \quad (55)$$

it follows that

$$\eta = -1 + 2\tau e^{-\tau^2} \int_0^{\tau} e^{\tau^2} d\tau. \quad (56)$$

The expressions (54) and (56) for the displacement of the point which was initially situated at the origin are equally inconvenient to discuss its subsequent motion, but it is not very difficult to obtain the general feature of it from the latter form. For the small value of τ the value of the function $e^{-\tau^2} \int_0^{\tau} e^{\tau^2} d\tau$ can be obtained by expanding the exponential function in the integral sign and integrating it term by term. For the large value of τ the asymptotic expression for the function $e^{-\tau^2} \int_0^{\tau} e^{\tau^2} d\tau$ is

$$\frac{1}{2\tau} + \frac{1}{2^2\tau^3} + \frac{1.3}{2^3\tau^5} + \frac{1.3.5}{2^4\tau^7} + \dots * \quad (57)$$

Therefore the course of the ordinate η at the origin tends approximately to the form

$$\eta = \frac{1}{2} \frac{4b}{gt^2} + \frac{1.3}{2^2} \left(\frac{4b}{gt^2}\right)^2 + \frac{1.3.5}{2^3} \left(\frac{4b}{gt^2}\right)^3 + \dots \quad (58)$$

when t is great.

* E. T. Whittaker, 'Modern Analysis,' p. 170; A. E. H. Love, 'Phil. Trans.,' A, vol. 207 (1907), p. 195.

Table I.

τ .	$e^{-\tau^2} \int_0^\tau e^{\tau^2} d\tau$.	$1 - 2\tau e^{-\tau^2} \int_0^\tau e^{\tau^2} d\tau$.	$\tau + (1 - 2\tau^2) e^{-\tau^2} \int_0^\tau e^{\tau^2} d\tau$.
0.0	0.00000	+1.00000	0.00000
0.1	0.09933	0.98013	+0.19744
0.2	0.19475	0.92210	0.37917
0.3	0.28263	0.83042	0.53176
0.4	0.35995	0.71204	0.64477
0.5	0.42443	0.57557	0.71222
0.6	0.47476	0.43029	0.73293
0.7	0.51034	0.28552	0.71021
0.8	0.53210	0.14864	0.65101
0.9	0.54073	+0.02669	0.56475
1.0	0.53808	-0.07616	0.46192
$\sqrt{2}$	0.4525	0.2799	+0.0567
$\sqrt{3}$	0.3640	0.2609	-0.0879
2	0.3013	0.2052	0.1091
$\sqrt{5}$	0.2585	0.1560	0.0904
$\sqrt{7}$	0.2075	0.0980	0.0517
3	0.1783	0.0698	0.031
5	0.1021*	0.0213†	0.0045‡
10	0.0503*	-0.0051†	-0.0005‡

* Calculated by using (57). † Calculated by using (58). ‡ Calculated by using (86).

By this Table the curve representing approximately the displacement at the origin can be easily traced, the unit of time and displacement being modified by (55).

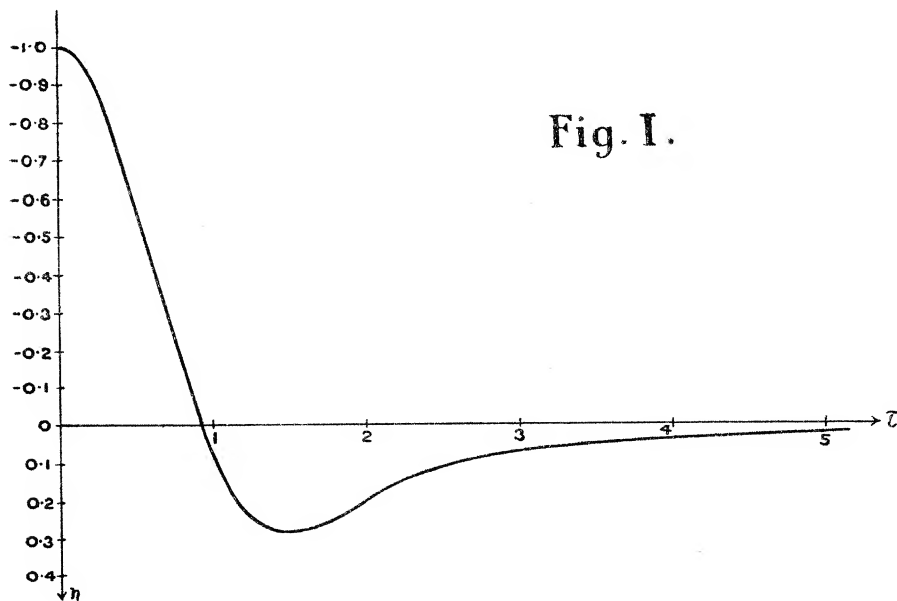


Fig. I.

The highest point in the curve corresponds to the initial prescribed elevation at the origin. As τ increases from zero the elevation decreases and reaches once to the value nil; and then begins the depression of the surface at the point under consideration. After that there is only one maximum of the depression, the amount of which is smaller than that of the initial elevation. Then it decreases more and more, very slowly, until, after an infinitely long time, it takes the limiting value zero. The number of zero-point and maximum-point can be determined without using the above figure.

We can assign several assumptions to the disturbing function to illustrate the movement at the origin. For example, if we take the form

$$f(\varpi) = Ab/(b^2 + \varpi^2)^{3/2} \quad (59)$$

instead of (48), then it appears that

$$\Phi(k) = Ae^{-bk}, \quad (60)$$

which can be found in a similar way as before.

In this case

$$\zeta = -A \int_0^\infty \cos \sigma t e^{-bk} J_0(k\varpi) k dk,$$

and, performing the integration, we have

$$\zeta = -\frac{A}{s_0^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n!}{(2n-2)!} \left(\frac{gt^2}{s_0}\right)^{n-1} P_n(\nu_0),^* \quad (61)$$

where s_0 and ν_0 are defined in (53).

At the origin this becomes

$$\zeta_0 = -\frac{A}{b^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n!}{(2n-2)!} \left(\frac{gt^2}{b}\right)^{n-1}. \quad (62)$$

Concerning this series and also as to the motion at the origin, we can make a similar interpretation to the former case.

II.

§ 5. Quite analogous treatments can be employed in the case where the initial condition is an impulse applied on the free surface. The typical solution is

$$\rho\phi = \cos \sigma t e^{-kz} J_m(k\varpi) \cos m\theta, \quad (63)$$

$$\zeta = 1/g\rho \cdot \sigma \sin \sigma t J_m(k\varpi) \cos m\theta. \quad (64)$$

[* This formula gives, in fact, the Cauchy-Poisson series as a first approximation when b is taken very small (Lamb's 'Hydrodynamics,' p. 408, equation 20). See the footnote on p. 59.]

Corresponding to the initial conditions

$$\rho\phi = F(\varpi) \cos m\theta, \quad \zeta = 0 \quad (65)$$

at the free surface, the general solution is

$$\phi = \frac{\cos m\theta}{\rho} \int_0^\infty \Psi(k) \cos \sigma t e^{-kz} J_m(k\varpi) k dk, \quad (66)$$

$$\zeta = \frac{\cos m\theta}{g\rho} \int_0^\infty \Psi(k) \sigma \sin \sigma t J_m(k\varpi) k dk; \quad (67)$$

where

$$\Psi(k) = \int_0^\infty F(\alpha) J_m(k\alpha) \alpha d\alpha. \quad (68)$$

At first we assume the initial impulse is given by

$$F(\varpi) = A/\varpi, \quad (69)$$

then the value of $\Psi(k)$ becomes

$$\Psi(k) = A/k,$$

corresponding to (16) in the former case.

The explicit form for the solution can be obtained by performing the operation $\frac{1}{g\rho} \frac{\partial}{\partial t}$ upon that of the former case. Thus from (24) and (25) we get

$$\begin{aligned} \phi = \frac{A \cos m\theta}{\rho r} \left\{ \sum_{n=0}^{m-1} (-1)^n \frac{(n+m)!}{2n!} \left(\frac{gt^2}{r} \right)^n P_n^{-m}(\mu) \right. \\ \left. + \sum_{n=m}^{\infty} (-1)^n \frac{(n-m)!}{2n!} \left(\frac{gt^2}{r} \right)^n P_n^m(\mu) \right\}, \quad (70) \end{aligned}$$

where

$$r = \sqrt{(z^2 + \varpi^2)}, \quad \mu = z/r;$$

and

$$\begin{aligned} \zeta = \frac{A \cos m\theta}{\rho g t \varpi} \left\{ \sum_{n=1}^{m-1} (-1)^{n-1} \frac{(n+m)!}{(2n-1)!} \left(\frac{gt^2}{\varpi} \right)^n P_n^{-m}(0) \right. \\ \left. + \sum_{n=m}^{\infty} (-1)^{n-1} \frac{(n-m)!}{(2n-1)!} \left(\frac{gt^2}{\varpi} \right)^n P_n^m(0) \right\}. \quad (71) \end{aligned}$$

Specially for $m = 0$, from (27) and (28),

$$\phi_{m=0} = \frac{A}{\rho r} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{2n!} \left(\frac{gt^2}{r} \right)^n P_n(\mu), \quad (72)$$

$$\zeta_{m=0} = \frac{A}{\rho g t \varpi} \left\{ \frac{1^2}{3!} \left(\frac{gt^2}{\varpi} \right)^2 - \frac{1^2 \cdot 3^2}{7!} \left(\frac{gt^2}{\varpi} \right)^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{11!} \left(\frac{gt^2}{\varpi} \right)^6 - \dots \right\}, \quad (73)$$

and for $m = 1$, from (29) and (30),

$$\phi_{m=1} = \frac{A \cos \theta}{\rho r} \left\{ P_0^{-1}(\mu) + \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{2n!} \left(\frac{gt^2}{r} \right)^n P_n^1(\mu) \right\}, \quad (74)$$

$$\zeta_{m=1} = \frac{A \cos \theta}{\rho g t \varpi} \left\{ \frac{1}{1!} \left(\frac{gt^2}{\varpi} \right) - \frac{1^2 \cdot 3}{5!} \left(\frac{gt^2}{\varpi} \right)^3 + \frac{1^2 \cdot 3^2 \cdot 5}{9!} \left(\frac{gt^2}{\varpi} \right)^5 - \dots \right\} \quad (75)$$

Now, when gt^2/ϖ is a large quantity, we can make use of the function T defined by (31). Thus

$$\zeta = - \frac{A \cos m\theta}{\rho g} \frac{\partial T_m(0)}{\partial t}.$$

Referring to the expressions for $T_0(0)$ and $T_1(0)$ found in (42) and (46), we get

$$\zeta_{m=0} = \frac{At}{\sqrt{2\rho\varpi^2}} \sin(gt^2/4\varpi) \quad (76)$$

for $m = 0$; and

$$\zeta_{m=1} = \frac{At}{\sqrt{2\rho\varpi^2}} \cos(gt^2/4\varpi) \cos \theta \quad (77)$$

for $m = 1$.

§ 6. As in § 4, we take the function expressing the initial impulse of the form

$$F(\varpi) = A/\sqrt{(b^2 + \varpi^2)} \quad (78)$$

to illustrate the history of the point which was situated initially at the origin.

For the case of symmetry about the origin ($m = 0$), the function $\Psi(k)$ becomes

$$\Psi(k) = Ae^{-bk}/k.$$

Thus we have

$$\phi = \frac{A}{\rho} \int_0^\infty \cos \sigma t e^{-(z+b)k} J_0(k\varpi) dk, \quad (79)$$

$$\zeta = \frac{A}{\rho g} \int_0^\infty \sigma \sin \sigma t \cdot e^{-bk} J_0(k\varpi) dk. \quad (80)$$

Or, performing the operation $\frac{1}{g\rho} \frac{\partial}{\partial t}$ upon (52) and (53), we obtain

$$\phi = \frac{A}{\rho s} \sum_{n=0}^\infty (-1)^n \frac{n!}{2n!} \left(\frac{gt^2}{s} \right)^n P_n(\nu), \quad (81)$$

$$\zeta = \frac{A}{\rho g t s_0} \sum_{n=1}^\infty (-1)^{n-1} \frac{n!}{(2n-1)!} \left(\frac{gt^2}{s_0} \right)^n P_n(\nu_0), \quad (82)$$

in which

$$s = \sqrt{(z+b)^2 + \varpi^2}, \quad \nu = (z+b)/s, \\ s_0 = \sqrt{(b^2 + \varpi^2)}, \quad \nu_0 = b/s_0$$

as before.

At the origin ($\varpi = 0$), ζ becomes

$$\zeta_0 = \frac{A}{\rho g t b} \sum_{n=1}^\infty (-1)^{n-1} \frac{n!}{(2n-1)!} \left(\frac{gt^2}{b} \right)^n. \quad (83)$$

This series can be put in a similar form to that in §4; or rather directly from (56) it follows that

$$\eta = \tau + (1 - 2\tau^2)e^{-\tau^2} \int_0^\tau e^{\tau^2} d\tau, \quad (84)$$

in which, for simplicity, it is put

$$\eta = (\rho b / A) \sqrt{(gb)} \zeta_0, \quad \tau = t \sqrt{(g/4b)}. \quad (85)$$

For large values of τ , using (57), we can take for η its asymptotic expansion of the form

$$-\frac{1}{2\tau^3} - \frac{2 \cdot 1 \cdot 3}{2^2 \tau^5} - \frac{3 \cdot 1 \cdot 3 \cdot 5}{2^3 \tau^7} - \frac{4 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2^4 \tau^9} - \dots \quad (86)$$

From these expressions we can make out the general feature of the movement of the point under consideration.

By the aid of the Table I we can draw the approximate diagram which represents the displacement at the origin, the units of scales being modified by (85).

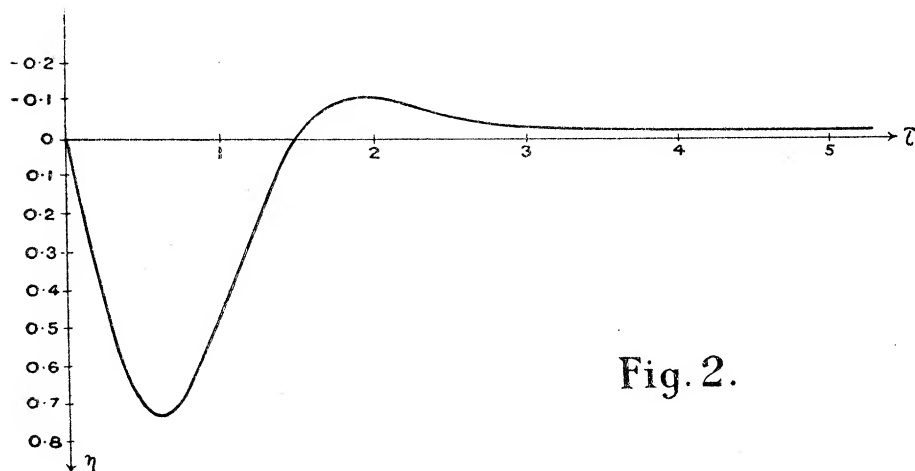


Fig. 2.

For $\tau = 0$, we have $\eta = 0$ and $d\eta/d\tau = 4$. Therefore as soon as the impulse is applied on the free surface the point at the origin gains suddenly a finite velocity and begins to move downwards. It will be seen, from the above figure, that at about $\tau = 0.6$ the depression becomes maximum, at about $\tau = 1.5$ it takes the value zero and then begins the elevation of the surface, and it takes the maximum value at about $\tau = 2$, after that it decreases slowly until it tends asymptotically to its limiting value zero, and the point comes to rest.

If we assign another assumption to the initial impulse like that expressed

in (59), it will probably take place by a similar mode of motion, but in a somewhat complicated form.

III.

§ 7. Now we come to the problem of surface waves caused by an explosion which takes place at a finite depth from the free surface.* In this case, we might take the value of the impulse, *i.e.* that of $\rho\phi$, given by the explosion as the initial datum, but the mathematics would be complicated and it might not give any concrete results. The following consideration, however, may serve to treat the problem without touching on the difficulties.

Suppose that an explosion under water sends out a pressural pulse of the type of a wave of sound, propagated by compression at high speed, and leaving unruffled water in its wake. As each ray reaches the free surface it is reflected totally, as required by the condition that the impulse is null at the surface, while a surface normal velocity caused by the pulse remains and determines a hydrodynamic flow throughout the water.† We may take the velocity of the propagation of the pressural pulse to be infinite, then the image method in electrostatics will be applicable as follows.

Let the point C at which the explosion occurs be at the depth h from the free surface and on the axis of z . Suppose the initial value of ϕ due to the source at C to be A/r , where r is the distance of any point in the water from C, then we must consider a fictitious negative source of equal strength at the image of C with regard to the free surface, to fulfil the condition that the initial impulse is zero at the free surface. The initial value of ϕ should then be

$$\phi = A(1/r - 1/r'),$$

$$\text{where} \quad r = \sqrt{(z-h)^2 + \varpi^2}, \quad r' = \sqrt{(z+h)^2 + \varpi^2}.$$

This value of ϕ gives the initial normal velocity

$$-\partial\phi/\partial z = -2Ah/(h^2 + \varpi^2)^{3/2} \quad (87)$$

over the free surface and it dies away of itself. Thus the problem is transformed to one of determining the surface motion, being given the initial surface normal velocity (87).

If we take

$$\phi = \cos \sigma t e^{-kz} J_0(k\varpi)$$

* This problem and the next following were undertaken mainly to illustrate in a general way the nature of the tidal waves produced by a submarine earthquake, with its source at a point or on a long straight fault in the strata.

† The source of the explosion is supposed to be so deep or its force so gentle that the surface of the water is not broken by the ejection of a column of water.

as the typical solution, then the general solution corresponding to the initial data (87) will be

$$\phi = \int_0^\infty \cos \sigma t e^{-kz} J_0(k\varpi) dk \int_0^\infty \left[-\frac{2Ah}{(h^2 + \alpha^2)^{3/2}} \right] J_0(k\alpha) \alpha d\alpha \quad (88)$$

with $\sigma = \sqrt{gk}$.

Since, by (60),

$$\int_0^\infty \frac{h}{(h^2 + \alpha^2)^{3/2}} J_0(k\alpha) \alpha d\alpha = e^{-hk},$$

we have
$$\phi = -2A \int_0^\infty \cos \sigma t e^{-(z+h)k} J_0(k\varpi) dk, \quad (89)$$

and accordingly
$$\zeta = -\frac{2A}{g} \int_0^\infty \sigma \sin \sigma t e^{-kh} J_0(k\varpi) dk. \quad (90)$$

These are the same expressions as (79) and (80) up to the constant factor. Therefore putting

$$s = \sqrt{(z+h)^2 + \varpi^2}, \quad \nu = (z+h)/s,$$

$$s_0 = \sqrt{h^2 + \varpi^2} \quad \nu_0 = h/s_0,$$

we get
$$\phi = -\frac{2A}{s} \sum_{n=0}^\infty (-1)^n \frac{n!}{2n!} \left(\frac{gt^2}{s} \right)^n P_n(\nu), \quad (91)$$

and
$$\zeta = -\frac{2A}{gts_0} \sum_{n=1}^\infty (-1)^{n-1} \frac{n!}{(2n-1)!} \left(\frac{gt^2}{s_0} \right)^n P_n(\nu_0). \quad (92)$$

At the point just above the explosion, that is at the origin, $\varpi = 0$, putting

$$gt^2/4h = \tau^2, \quad (93)$$

we have
$$\zeta_0 = -\frac{2A}{\sqrt{gh^3}} \left\{ \tau + (1-2\tau^2)e^{-\tau^2} \int_0^\tau e^{\tau'^2} d\tau' \right\}. \quad (94)$$

Comparing (94) with (84), we see that the curve representing the displacement at the origin is quite the same as in fig. (2), only inverted in form and the unit of ordinate being different.

If we take

$$\tau_1 = 0.6, \quad \tau_2 = 1.5, \quad \tau_3 = 2,$$

as the values of τ which give the maximum, zero and the minimum of the displacement, and $g = 32$, then the times at which they occur will be given by

$$t_1 = \frac{3}{10} \sqrt{\frac{h}{2}}, \quad t_2 = \frac{1.5}{2} \sqrt{\frac{h}{2}}, \quad t_3 = \sqrt{\frac{h}{2}},$$

respectively, in which t_1 , t_2 , and t_3 are expressed in seconds and h in feet. Conversely, if we can measure any of them, then the depth h can be calculated approximately by using the above relations.

§ 8. We will conclude this paper by discussing the explosion problem in a horizontal canal of infinite depth and infinite length. Take the axis of x along the canal and on the undisturbed free surface of water, and that of z vertical downwards as before. If we suppose that all the circumstances are independent of y , we must consider the initial value of ϕ given by the explosion to be of the form $-A \log r$.^{*} Therefore, considering the image as before, we have the effective initial normal velocity at the free surface

$$-\partial\phi/\partial z = -2Ah/(h^2 + x^2) \quad (95)$$

in place of (87). Corresponding to this initial velocity, the general solution will be

$$\phi = \frac{2}{\pi} \int_0^\infty \cos \sigma t e^{-kz} \cos kx \frac{dk}{k} \int_0^\infty \left[-\frac{2Ah}{h^2 + \alpha^2} \right] \cos k\alpha d\alpha,$$

in which $\sigma = \sqrt{gk}$.

By a well known formula

$$\int_0^\infty \frac{h \cos k\alpha}{h^2 + \alpha^2} d\alpha = \frac{1}{2} \pi e^{-hk}, \dagger$$

we get
$$\phi = -2A \int_0^\infty \cos \sigma t e^{-(z+h)k} \cos kx dk/k,$$

and consequently
$$\zeta = -\frac{2A}{g} \int_0^\infty \frac{\sigma \sin \sigma t}{k} e^{-hk} \cos kx dk. \quad (96)$$

Expanding $\sin \sigma t$ and then making use of the integral formulæ (34), it may be shown that

$$\zeta = -\frac{2At}{g} \sum_{n=0}^\infty (-1)^n \frac{n!}{(2n+1)!} \left(\frac{gt^2}{g} \right)^n \cos(n+1)\psi, \quad (97)$$

where $g = \sqrt{(x^2 + h^2)}, \quad \psi = \tan^{-1}(x/h).$

At the point just above the explosion, *i.e.* along the axis of y , $x = 0$,

$$\zeta_0 = -\frac{2At}{h} \sum_{n=0}^\infty (-1)^n \frac{n!}{(2n+1)!} \left(\frac{gt^2}{h} \right)^n.$$

This series can be put in the form

$$\zeta_0 = -\frac{4A}{\sqrt{gh}} e^{-\tau^2} \int_0^\tau e^{\tau'^2} d\tau', \quad (98)$$

in which τ is defined by (93).

^{*} Since this work was finished my attention has been called to a paper by Prof. Lamb in the 'Annali di Matematica,' vol. 21, p. 237 (1913), where the surface waves that accompany a cylinder travelling uniformly transverse to its length under water are investigated.

[†] H. Weber, 'Die part. Diff.-Gleich.,' vol. 1, p. 43.

For large values of t , if we use the asymptotic expansion (57), then

$$\zeta_0 = -\frac{4A}{gt} \left\{ 1 + 1 \left(\frac{2h}{gt^2} \right) + 1.3 \left(\frac{2h}{gt^2} \right)^2 + 1.3.5 \left(\frac{2h}{gt^2} \right)^3 + \dots \right\}.$$

Making use of Table I we can trace the curve for ζ_0 .

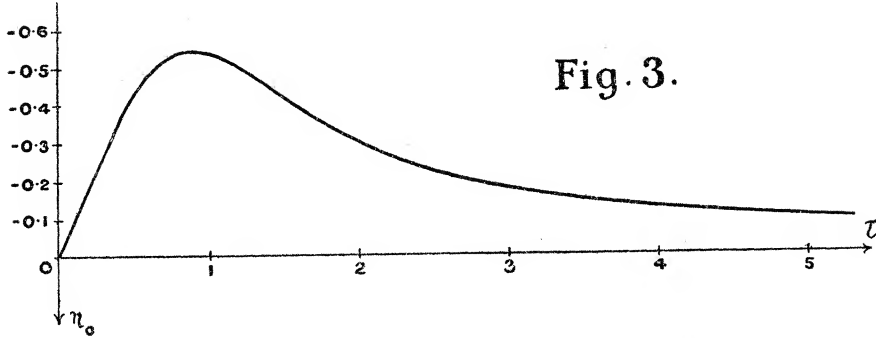


Fig. 3.

In this figure the units of length and time are modified by

$$\zeta_0 = 4A\eta_0/\sqrt{gh}, \quad gt^2/4h = \tau^2.$$

From this figure it will be seen that the line which was initially situated along the axis of y is suddenly put in motion and there begins an elevation of the free surface, and that after a certain time (about $\tau = 0.9$) there is the one only maximum of the elevation, and then it decreases more and more until the surface takes the original plane form.

If we take $\tau = 0.9$ for the value of τ which gives the maximum of the elevation, then the time at which it occurs is given by

$$t = 0.45\sqrt{(\frac{1}{2}h)},$$

in which t is reckoned from the time of explosion in seconds and h is measured by feet.

Lastly, at a point very far from the source of disturbance, if we neglect h compared with x , we may put $q = x$ and $\psi = \pi/2$ in (97). Then we have

$$\zeta_\infty = -\frac{2At}{x} \left\{ \frac{1}{1.3} \left(\frac{gt^2}{2x} \right) - \frac{1}{1.3.5.7} \left(\frac{gt^2}{2x} \right)^3 + \frac{1}{1.3.5.7.9.11} \left(\frac{gt^2}{2x} \right)^5 - \dots \right\} * \quad (99)$$

* This formula and those that follow are applicable only when $gt^2/2x$ is moderately small, this being the only range of values which matters in the physical problem, each phase of the physical disturbance travelling, as is well known, with constant acceleration. When $gt^2/2x$ is large, the most essential terms of the series (99) are situated far along it, and in deducing them from (97) we must not replace $\cos(n+1)\psi$ by unity. Thus the impossible physical feature of persistence of the amplitude for all time with period diminishing rapidly without limit, which is indicated in (100) and in fig. 4, would not occur in an exact solution: it can be cut away while the earlier undulations of the curve remain applicable.

or referring to (35) and (36) this can be put in the form

$$\zeta_{\infty} = -2At/x \cdot N(\omega)/\omega$$

with $\omega = gt^2/2x$.

For large values of $gt^2/2x$, from (38), we get

$$\zeta_{\infty} = -A\sqrt{\left(\frac{2\pi}{gx}\right)} \left\{ \sin \frac{gt^2}{4x} - \cos \frac{gt^2}{4x} \right\} - \frac{4A}{gt} \left\{ 1 - 1.3 \left(\frac{2x}{gt^2}\right)^2 + \dots \right\}.* \quad (100)$$

Further, if we put

$$\eta_{\infty} = \frac{1}{2A} \sqrt{(gx/2)} \zeta_{\infty}, \quad t' = t\sqrt{(g/2x)},$$

then

$$\eta_{\infty} = -N(t'^2)/t'. \quad (101)$$

Again, if we put

$$\xi_{\infty} = gt\zeta_{\infty}/4A, \quad x' = 2x/gt^2,$$

then we have

$$\xi_{\infty} = -N(1/x'). \quad (102)$$

The function $N(z)/z$ has been tabulated by E. Lommel in his memoir on the diffraction problem,[†] and the other functions needed are deduced in Table II, so that the forms of the few waves can be traced without difficulty.

Table II.

z .	$\frac{N(z)}{z}$.	$\frac{N(z)}{\sqrt{z}}$.	$N(z)$.
0	0.000000	0.00000	0.00000
1	+0.323905	+0.32391	+0.32391
2	0.593492	0.83933	1.18698
3	0.765194	1.32535	2.29558
4	0.814623	1.62925	3.25849
5	0.741108	1.65717	3.70554
6	0.567111	1.38913	3.40267
7	0.333027	0.88110	2.33119
8	+0.088400	+0.25008	+0.70720
9	-0.118539	-0.35562	-1.06685
10	0.250619	0.79253	2.50619
11	0.288825	0.95793	3.17708
12	0.235250	0.81493	2.82300
13	-0.111413	-0.40171	-1.44837
14	+0.047520	+0.17780	+0.66528
15	0.201198	0.77924	3.01797

* The first term corresponds to the known Cauchy-Poisson expression for the disturbance at a distance due to a point source, and the second is the modification in it that is produced by the finite extent of the source: it represents a gradual fall of the surface to the general level.

† 'Abh. d. k. Bayer. Akad. d. Wiss., 2te Cl.,' vol. 15 (1886), Tafel V.

Table II—*continued.*

$z.$	$\frac{N(z)}{z.}$	$\frac{N(z)}{\sqrt{z.}}$	$N(z).$
16	0·313279	1·25312	5·01246
17	0·359296	1·48142	6·10803
18	0·331493	1·40641	5·96687
19	0·239689	1·04478	4·55409
20	+0·108121	+0·48353	+2·16242
21	−0·030844	−0·14134	−0·64772
22	0·144587	0·67817	3·18091
23	0·207839	0·99676	4·78030
24	0·208258	1·02025	4·99819
25	0·148793	0·74397	3·71982
26	−0·046392	−0·23655	−1·24620
27	+0·072514	+0·37679	+1·95788
28	0·178590	0·94501	5·00052
29	0·246628	1·32844	7·14221
30	0·261363	1·43138	7·84089
31	0·220766	1·22912	6·84375
32	0·136088	0·76983	4·35482
33	+0·028774	+0·16529	+0·94954
34	−0·074961	−0·43709	−2·54867
35	0·150517	0·83103	5·26810
36	0·180741	1·08444	6·50668
37	0·159812	0·97210	5·91304
38	0·094332	0·58150	3·58462
39	−0·001410	−0·00881	−0·05499
40	+0·095698	+0·60525	+3·82792
41	0·173327	1·10984	7·10641
42	0·213082	1·38093	8·94944
43	0·206133	1·35170	8·86372
44	0·155109	1·02888	6·82480
45	+0·073202	+0·49105	+3·29409
46	−0·019241	−0·13050	−0·88509
47	0·099771	0·68400	4·68924
48	0·149290	1·03431	7·16592
49	0·156580	1·09606	7·67242
50	0·120841	0·85448	6·04205
51	−0·051678	−0·36906	−2·63558
52	+0·033421	+0·24100	+1·73789
53	0·113462	0·82602	6·01349
54	0·169071	1·24242	9·12983
55	0·187150	1·38795	10·29325
56	0·163922	1·22668	9·17963
57	0·105670	0·79779	6·02319
58	+0·027050	+0·20605	+1·56890
59	−0·052604	−0·40406	−3·10364
60	0·114037	0·88333	6·84222

The annexed curves are the graphical representations of equations (101) and (102), which are traced by using the above Table. They enable us to find

out how the displacement of the surface at a great distance from the source of the explosion varies for different times and different places respectively.

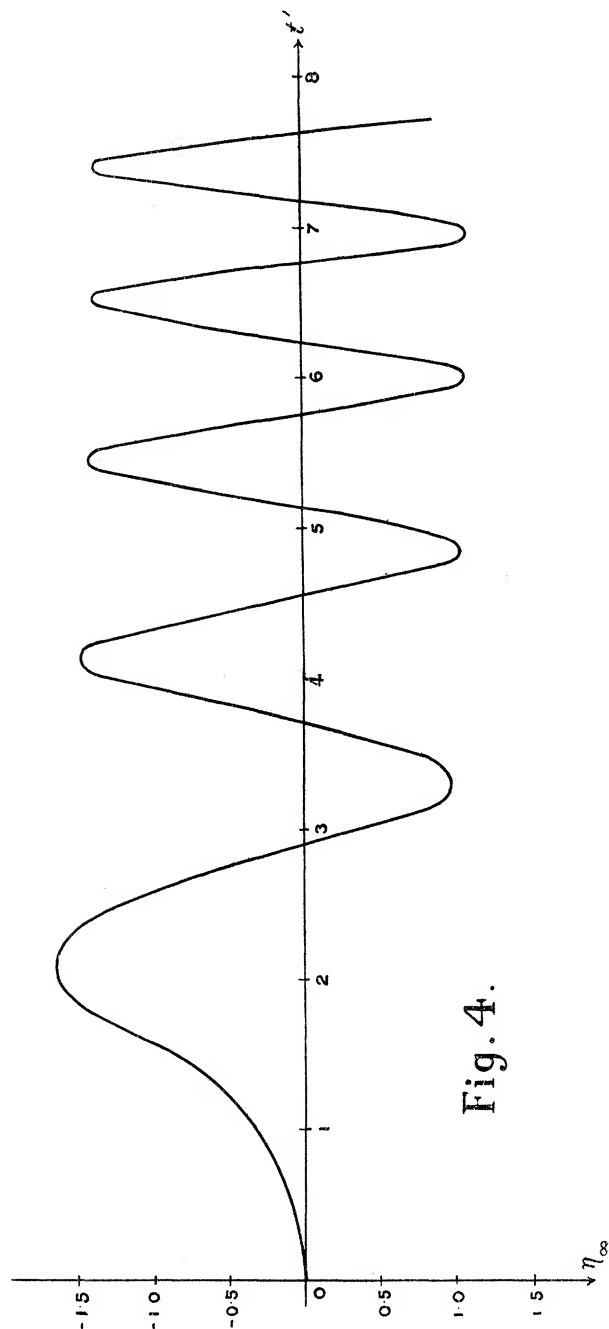


Fig. 4.

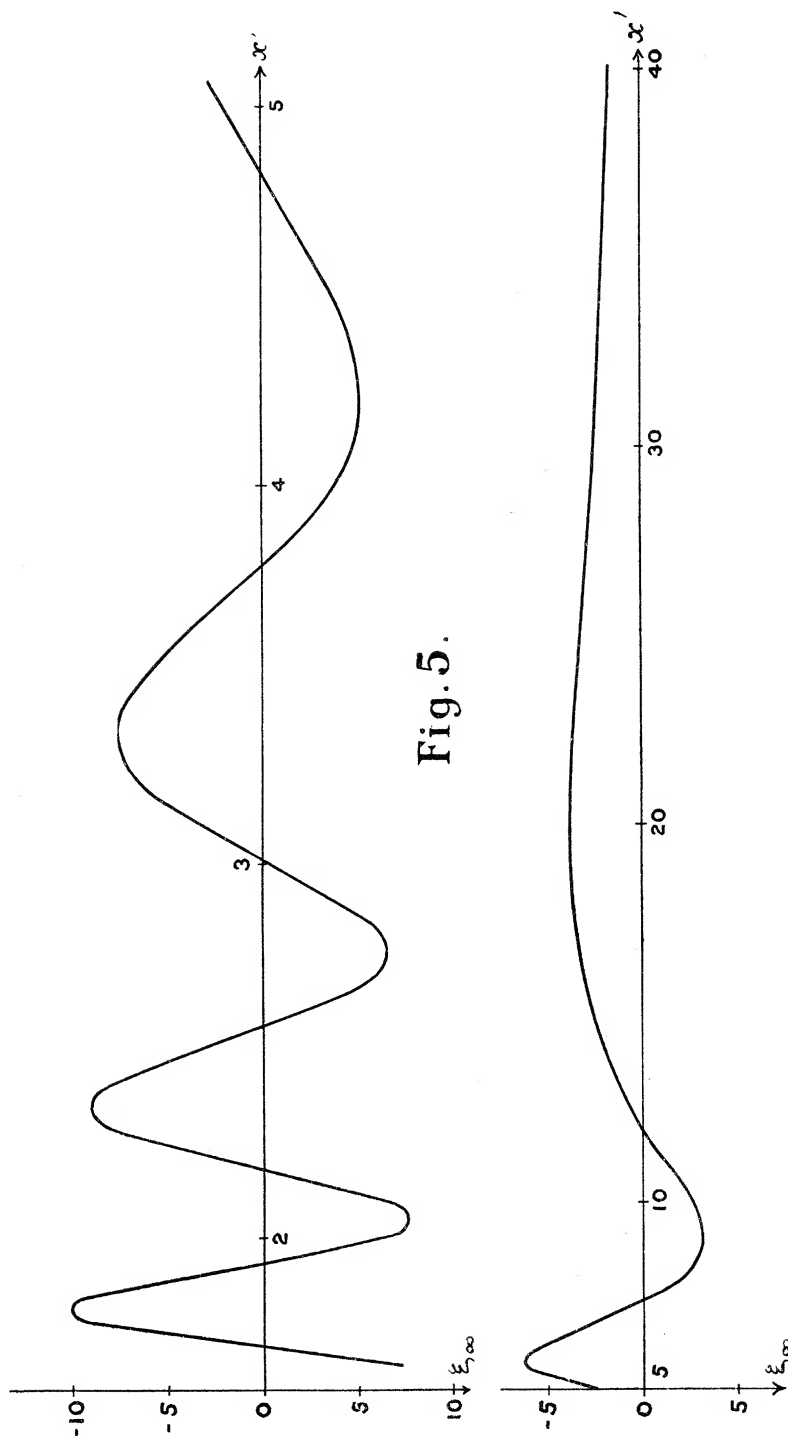


Fig. 5.

The figure 4, which belongs to equation (101), shows the rise and fall of the surface at various times and at a particular place of a great distance from the origin, the unit of time being modified by $t' = t\sqrt{(g/2x)}$. For moderately small values of t' , which alone are applicable, the curve is unsymmetrical with regard to the axis of time and as the time increases the rise and fall becomes more and more symmetrical and, on the other hand, the oscillations follow one another with ever decreasing period. For larger values of t' , the course of the curve ceases to represent the motion, which really dies away, for reasons explained already (foot note *supra*).

As a practical application, the earlier part of this curve might serve to determine the distance of the point under observation from the source of the explosion, if we measure the time of any zero-point of the displacement or the interval (*i.e.* period) between any two adjacent ones. For example, if we take $t' = 2.9, 3.7$, as the values of t' of the zero-points, and if we measure the actual time θ between them, then the distance x will be given by

$$x = g\theta^2/2(3.7 - 2.9)^2 = 25\theta^2$$

in the foot-second-unit system. It will be noticed that the period θ in this formula would be a large quantity of the order $\sqrt{x/5}$.

The figure 5, which belongs to the formula (102), shows the wave profile at a particular time at various distances from the origin, the unit of horizontal scale being modified by $x' = (2x/gt^2) \times 100$. The second part of the diagram is condensed along the axis of x' to 1/10 scale of the first part. As we advance farther and farther towards the infinite distant point from the origin, the essential groups of the waves are found to increase continually in length and to diminish continually in height, in such a manner that t^2/x remains constant in each phase. This figure applies only at great distances from the origin, *i.e.* when x is large, so that t has to be large for the earlier part to be applicable; *e.g.*, $x' = 2$ gives $t = 1.8\sqrt{x}$ in the foot-second-unit system.

In the region some distance away where equation (102) holds, and therefore the fig. 5 is applicable, the wave-length can be determined by finding the values of x' of the earlier zero-points in this diagram.

In conclusion I wish to express my sincere thanks to Prof. Sir J. Larmor for his kind advice during the progress of this work.

